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On nesting of G-decompositions of λK_v where G has four nonisolated vertices or less ☆

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Abstract

The complete multigraph λK_v is said to have a G-decomposition if it is the union of edge disjoint subgraphs of K_v each of them isomorphic to a fixed graph G. The spectrum problem for G-decompositions of λK_v that have a nesting was first considered in the case $G = K_3$ by Colbourn and Colbourn (Ars Combin. 16 (1983) 27–34) and Stinson (Graphs and Combin. 1 (1985) 189–191). For $\lambda = 1$ and $G = C_m$ (the cycle of length m) this problem was studied in many papers, see Lindner and Rodger (in: J.H. Dinitz, D.R. Stinson (Eds.), Contemporary Design Theory: A Collection of Surveys, Wiley, New York, 1992, p. 325–369), Lindner et al. (Discrete Math. 77 (1989) 191–203), Lindner and Stinson (J. Combin. Math. Combin. Comput. 8 (1990) 147–157) for more details and references. For $\lambda = 1$ and $G = P_k$ (the path of length $k-1$) the analogous problem was considered in Milici and Quattrocchi (J. Combin. Math. Combin. Comput. 32 (2000) 115–127). In this paper we solve the spectrum problem of nested G-decompositions of λK_v for all the graphs G having four nonisolated vertices or less, leaving eight possible exceptions.

1. Introduction

Let $H = (V(H), E(H))$ be a graph. Denote by λH the graph H in which every edge has multiplicity λ. The multigraph λH is said to be *G-decomposable* if it is a union of edge disjoint subgraphs of K_v, each of them isomorphic to a fixed graph G. This

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situation is denoted by $\lambda H \to G$; λH is also said to admit a *G-decomposition* $(V, \mathscr{B})$, where $V = V(H)$, the vertex set of H, and $\mathscr{B}$ is the edge-disjoint decomposition of λH into copies of G. Usually $\mathscr{B}$ is called the *block-set* of the G-design and any $B \in \mathscr{B}$ is said to be a *block*.

A G-decomposition of λK_v is also called a *G-design of order v, block size* $|V(G)|$ *and index* λ. A G-design $(W, \mathscr{A})$ is called to be a *subdesign* of $(V, \mathscr{B})$ if $W \subseteq V$ and $\mathscr{A} \subseteq \mathscr{B}$.

The set of values of v for which K_v has a G-decomposition is determined if G has four vertices or less [2].

Let *m-star* be the graph $S_m = [a; a_1, a_2, \ldots, a_m] = \{\{a, a_1\}, \{a, a_2\}, \ldots, \{a, a_m\}\}$. The vertex a of degree m in S_m is called the *centre* of the star and the vertices a_i of degree 1 are called the *terminal vertices* of the star. Tarsi [17] found necessary and sufficient conditions for the existence of a $\lambda K_v \to S_m$.

An *m-cycle system* of order v and index λ (mCS_λ) is a C_m-design of λK_v, where C_m is an *m-cycle* $(a_1, a_2, \ldots, a_m) = \{\{a_1, a_2,\}, \{a_2, a_3\}, \ldots, \{a_{m-1}, a_m\}, \{a_1, a_m\}\}$. We will denote by mCS an mCS_1.

Definition (*see Lindner and Rodger* [9], *Milici and Quattrocchi* [13]). A *nesting of a G-decomposition of* λH $(V, \mathscr{B})$ is a pair $\{(V, \mathscr{S}), F\}$ where $(V, \mathscr{S})$ is a $\lambda H \to S_m$ and $F : \mathscr{B} \to \mathscr{S}$ is a $1-1$ mapping such that:

(n_1) for every $B \in \mathscr{B}$ the centre of the m-star $F(B)$ is not in $V(B)$ and any terminal vertex of $F(B)$ is in $V(B)$;
(n_2) For every pair $B_1, B_2 \in \mathscr{B}$ the graphs $B_1 \cup F(B_1)$ and $B_2 \cup F(B_2)$ are isomorphic.

Example 1. Let $V(K_9) = Z_9$. For $i \in Z_9$ put $B^i = (i, 1+i, 7+i, 2+i)$ and $S^i = [3+i; i, 1+i, 7+i, 2+i]$, reducing all sums modulo 9. Then $(Z_9, \{B^i \mid i \in Z_9\})$ is a $4CS$ of order 9 that has a nesting defined by $(Z_9, \{S^i \mid i \in Z_9\})$ and $F(B^i) = S^i$.

Let $\hat{G} = G \cup S_m$, where the centre of S_m is not in $V(G)$ and any terminal vertex of S_m is in $V(G)$. It is clear that a nesting of a G-decomposition of λH is a $2\lambda H \to \hat{G}(V(H), \mathscr{N})$ such that:

(p_1) $(V(H), \{B_1 \mid B \in \mathscr{N}\})$ (where B_1 is the subgraph of B isomorphic to G) is a decomposition $\lambda H \to G$;
(p_2) $(V(H), \{B_2 \mid B \in \mathscr{N}\})$ (where B_2 is the subgraph of B isomorphic to S_m) is a decomposition $\lambda H \to S_m$.

When $H = K_v$, we say that the nested G-decomposition of λK_v, $2\lambda K_v \to \hat{G}$, is a *$\hat{G}$-design* $N(v, 2\lambda)$.

The spectrum problem for $\widehat{C_m}$*-designs* $N(v, 2\lambda)$ was first considered in the case where $m = 3$ by Colbourn and Colbourn [4] and Stinson [15]. For $\lambda = 1$ this problem was studied by Lindner et al. [10] for odd m and by Lindner and Stinson [11] for even m. See also [9] for more details and references. In the following theorem we state the known results for $m = 3, 4$.

Theorem 1 (Colbourn and Colbourn [4] and Stinson [15,16]). *There exists a $\widehat{C_3}$-designs $N(v,2\lambda)$ if and only if $\lambda(v-1)\equiv 0 \pmod 6$, $v\geqslant 4$. For every $v\equiv 1 \pmod 8$ there is a $\widehat{C_4}$-designs $N(v,2)$ except possibly if $v\in\{57,65,97,113,185,265\}$.*

Direct constructions of $\widehat{C_4}$-designs of order 65, 97 and 113 are given in [14].

In this paper, we consider the case where G is a graph with four nonisolated vertices or less and we solve the spectra problem of nested G-decompositions of λK_v, except possibly for eight values of v.

It will cause no confusion if a graph G identified by its edge set $E(G)$, since no graphs have isolated vertices.

2. Preliminaries

In this section, we shall define some terminology and state some results which will be useful later on.

Lemma 1. *Let $(V,\mathscr{B})$ be a G-decomposition of λK_v, $v>1$, having a nesting $\{(V,\mathscr{S}), F\}$. Then the following conditions hold (1) $v\geqslant 1+|V(G)|$; (2) $\lambda v(v-1)\equiv 0 \pmod{2\,|E(G)|}$; (3) $\lambda(v-1)\equiv 0 \pmod{\alpha(G)}$, where $\alpha(G)$ is the greatest common divisor of the degrees of the vertices of G; (4) $|E(G)|=|V(S_m)|-1=m$; and (5) $|E(G)|\leqslant|V(G)|$.*

Proof. Conditions (1)–(3) are straightforward. Condition (4) follows from the equality $|\mathscr{B}|=|\mathscr{S}|$. To prove (5) note that $|E(G)|+1=m+1=|V(S_m)|\leqslant|V(G)|+1$. $\square$

By Lemma 1 there is not a nested G-decomposition of λK_v for $G=K_4$ and $G=K_4-e$ (the quadrilateral with one diagonal). Since the spectrum problem for nested K_3-decompositions of λK_v is solved (see Theorem 1), the following cases must be considered:

(i) $G=P_2=[a_1,a_2]=\{a_1,a_2\}$, the path of length 1.
(ii) $G=P_3=S_2=[a_1,a_2,a_3]=[a_2;a_1,a_3]=\{\{a_1,a_2\},\{a_2,a_3\}\}$, the path of length 2 or the 2-star of centre a_2.
(iii) $G=E_2=[a_1,a_2;a_3,a_4]=\{\{a_1,a_2\},\{a_3,a_4\}\}$, two edges having no common vertex.
(iv) $G=S_3=[a;a_1,a_2,a_3]=\{\{a,a_1\},\{a,a_2\},\{a,a_3\}\}$, the 3-star of centre a.
(v) $G=P_4=[a_1,a_2,a_3,a_4]=\{\{a_1,a_2\},\{a_2,a_3\},\{a_3,a_4\}\}$, the path of length 3.
(vi) $G=D=[a_1,a_2,a_3\bowtie a_4]=\{\{a_1,a_2\},\{a_1,a_3\},\{a_2,a_3\},\{a_3,a_4\}\}$, the triangle with attached edge .
(vii) $G=C_4=(a_1,a_2,a_3,a_4)=\{\{a_1,a_2\},\{a_2,a_3\},\{a_3,a_4\},\{a_4,a_1\}\}$, the cycle of length 4.

Let $(V,\mathscr{B})$ be a $\hat{G}$-*design* $N(v,2\lambda)$. If $|E(G)|<|V(G)|$ then every block $B\in\mathscr{B}$ contains exactly $|V(G)|-|E(G)|$ vertices missing on the vertex set of $F(B)$. So, to satisfy

(n_2) of Definition, it is necessary to decide the *position* of these vertices into the block B.

Example 2. (2.1) $(Z_9, \{Q^i = [i, 3+i, 7+i \bowtie 8+i] \mid i \in Z_9\})$ is a D-decomposition of K_9 that has a nesting defined by $(Z_9, \{S^i = [2+i; i, 3+i, 7+i, 8+i] \mid i \in Z_9\})$, reducing all the sums modulo 9, and $F(Q^i) = S^i$.

(2.2) $(Z_5, \{Q_1^i = [i, 1+i, 3+i, 2+i], Q_2^i = [3+i, 4+i, 2+i, i] \mid i \in Z_5\})$ is a P_4-decomposition of $3K_5$ that has a nesting defined by $(Z_5, \{S_1^i = [4+i; i, 1+i, 3+i], S_2^i = [1+i; 3+i, 4+i, 2+i] \mid i \in Z_5\})$ and $F(Q_\rho^i) = S_\rho^i$, $\rho = 1, 2$.

(2.3) $(Z_7, \{Q^i = [5+i, 1+i, 6+i, i] \mid i \in Z_7\})$ is a P_4-decomposition of K_7 that has a nesting defined by $(Z_7, \{S^i = [4+i; 5+i, 6+i, i] \mid i \in Z_7\})$ and $F(Q^i) = S^i$.

(2.4) Let $Q_1^i = [(i,0); (1+i,0), (2+i,0), (i,1)]$, $Q_2^i = [(i,0); (1+i,1), (2+i,1), (3+i,1)]$, $Q_3^i = [(i,1); (1+i,0), (1+i,1), (3+i,1)]$, $Q_4^i = [(i,1); (i,2), (1+i,2), (2+i,2)]$, $Q_5^i = [(i,2); (1+i,2), (3+i,2), (i,0)]$, $Q_6^i = [(i,0); (1+i,2), (2+i,2), (3+i,2)]$, $Q_7^i = [(i,2); (1+i,0), (2+i,1), (1+i,1)]$, $S_1^i = [(3+i,0); (1+i,0), (2+i,0), (i,1)]$, $S_2^i = [(2+i,0); (1+i,1), (2+i,1), (3+i,1)]$, $S_3^i = [(4+i,1); (1+i,0), (1+i,1), (3+i,1)]$, $S_4^i = [(1+i,1); (i,2), (1+i,2), (2+i,2)]$, $S_5^i = [(4+i,2); (1+i,2), (3+i,2), (i,0)]$, $S_6^i = [(1+i,0); (1+i,2), (2+i,2), (3+i,2)]$ and $S_7^i = [(4+i,2); (1+i,0), (2+i,1), (1+i,1)]$, where i is in Z_5 and the sum is (mod 5). Then $(Z_5 \times Z_3, \{Q_\rho^i \mid i \in Z_5, \rho = 1, 2, \ldots, 7\})$ is a S_3-decomposition of K_{15} that has a nesting defined by $(Z_5 \times Z_3, \{S_\rho^i \in Z_5, \rho = 1, 2, \ldots, 7\})$ and $F(Q_\rho^i) = S_\rho^i$.

(2.5) Let $Q_1^i = [i; \infty, 1+i, 3+i]$, $Q_2^i = [i; \infty, 1+i, 2+i]$, $Q_3^i = [i; \infty, 3+i, 2+i]$, $Q_4^i = [i; 1+i, 2+i, 4+i]$, $S_1^i = [2+i; \infty, i, 1+i]$, $S_2^i = [3+i; \infty, i, 1+i]$, $S_3^i = [4+i; \infty, i, 3+i]$ and $S_4^i = [3+i; 1+i, i, 2+i]$, where i is in Z_7 and the sum is (mod 7). Then $(Z_7 \cup \{\infty\}, \{Q_\rho^i \mid i \in Z_7, \rho = 1, 2, 3, 4\})$ is a S_3-decomposition of $3K_8$ that has a nesting defined by $(Z_7 \cup \{\infty\}, \{S_\rho^i \mid i \in Z_7, \rho = 1, 2, 3, 4\})$ and $F(Q_\rho^i) = S_\rho^i$.

(2.6) $(Z_7 \cup \{\infty\}, \{Q_1^i = [\infty, 2+i; 1+i, 4+i], Q_2^i = [i, 1+i; 5+i, 3+i] \mid i \in Z_7\})$ is an E_2-decomposition of K_8 that has a nesting defined by $(Z_7 \cup \{\infty\}, \{S_1^i = [i; \infty, 1+i], S_2^i = [2+i; i, 5+i] \mid i \in Z_7\})$ and $F(Q_\rho^i) = S_\rho^i$, $\rho = 1, 2$.

(2.7) $(Z_5, \{Q_1^i = [i, 1+i; 2+i, 4+i], Q_2^i = [2+i, 4+i; i, 1+i] \mid i \in Z_5\})$ is an E_2-decomposition of $2K_5$ that has a nesting defined by $(Z_5, \{S_1^i = [3+i; i, 1+i], S_2^i = [3+i; 2+i, 4+i] \mid i \in Z_5\})$ and $F(Q_\rho^i) = S_\rho^i$, $\rho = 1, 2$.

To denote the graph $\hat{G}$ we will use the following notation (the symbol $\hat{x}$ means that x is a vertex of G missing on the vertex set of S_m):

(b$_1$) For $G = P_2 = [a_1, a_2]$ it is $\hat{G} = \widehat{P_2} = \langle a_1, \widehat{a_2}; a\rangle = [a_1, a_2] \cup [a; a_1]$ (see Fig. 1).

(b$_2$) For $G = E_2 = [a_1, a_2; a_3, a_4]$ it is either $\hat{G} = \widehat{E_2} = \langle a_1, \widehat{a_2}; a_3, \widehat{a_4}; a\rangle = [a_1, a_2; a_3, a_4] \cup [a; a_1, a_3]$ or $\hat{G} = \widehat{\widehat{E_2}} = \langle a_1, a_2; \widehat{a_3}, \widehat{a_4}; a\rangle = [a_1, a_2; a_3, a_4] \cup [a; a_1, a_2]$ (see Fig. 2).

(b$_3$) For $G = S_3 = [a; a_1, a_2, a_3]$ it is either $\hat{G} = \widehat{S_3} = \langle \hat{a}; a_1, a_2, a_3; c\rangle = [a; a_1, a_2, a_3] \cup [c; a_1, a_2, a_3]$ or $\hat{G} = \widehat{\widehat{S_3}} = \langle a; a_1, a_2, \widehat{a_3}; c\rangle = [a; a_1, a_2, a_3] \cup [c; a, a_1, a_2]$ (see Fig. 3).

(b$_4$) For $G = P_k = [a_1, a_2, \ldots, a_k]$ $(k = 3, 4)$, it is either $\hat{G} = \widehat{P_k} = \langle a_1, a_2, \ldots, \widehat{a_k}; a\rangle = [a_1, a_2, \ldots, a_k] \cup [a; a_1, a_2, \ldots, a_{k-1}]$ or $\hat{G} = \widehat{\widehat{P_k}} = \langle a_1, \widehat{a_2}, \ldots, a_k; a\rangle = [a_1, a_2, \ldots, a_k] \cup [a; a_1, a_3, \ldots, a_k]$ (see Fig. 4).

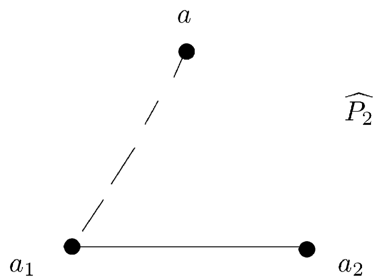

Fig. 1.

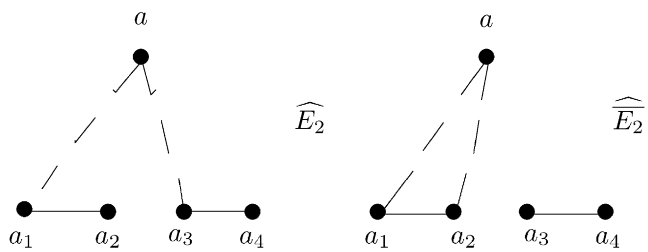

Fig. 2.

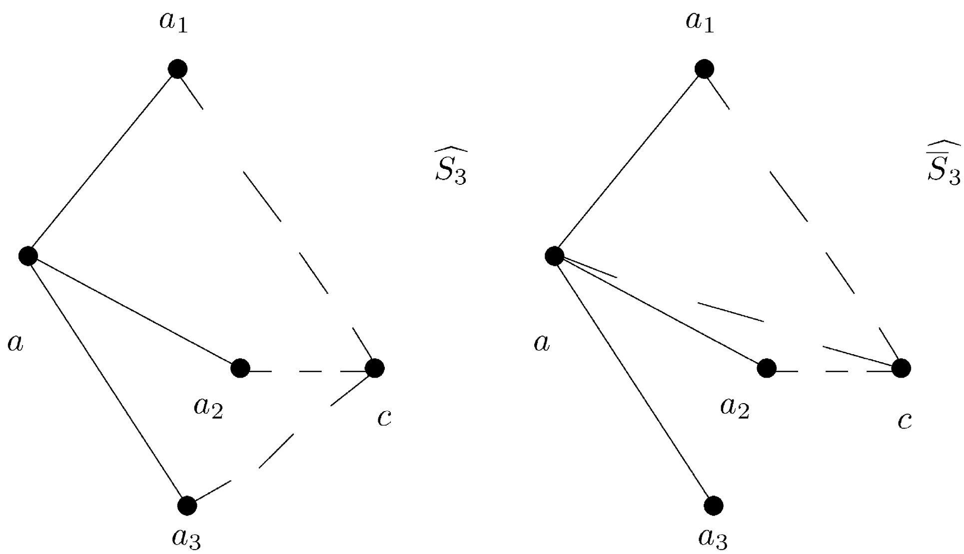

Fig. 3.

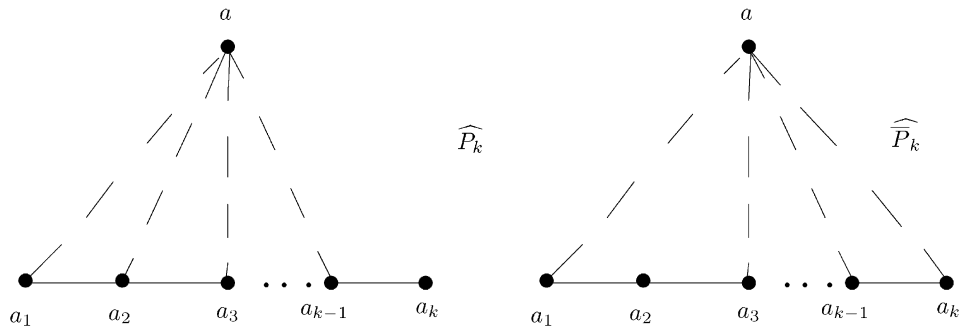

Fig. 4.

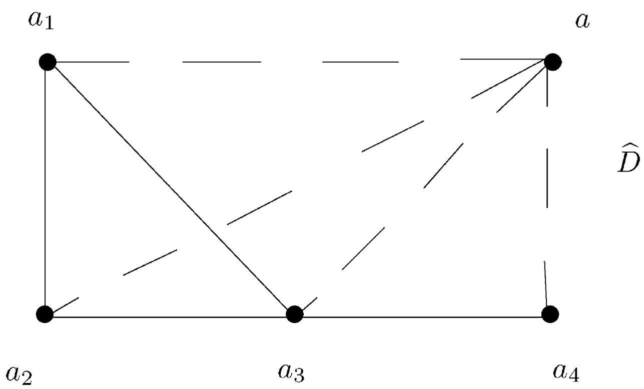

Fig. 5.

(b_5) For $G = D = [a_1, a_2, a_3 \bowtie a_4]$, it is $\hat{G} = \hat{D} = \langle a_1, a_2, a_3 \bowtie a_4; a\rangle = [a_1, a_2, a_3 \bowtie a_4] \cup [a; a_1, a_2, a_3, a_4]$ (see Fig. 5).
(b_6) For $G = C_4 = (a_1, a_2, a_3, a_4)$, it is $\hat{G} = \widehat{C_4} = \langle a_1, a_2, a_3, a_4; a\rangle = (a_1, a_2, a_3, a_4) \cup [a; a_1, a_2, a_3, a_4]$ (see Fig. 6).

The authors studied in [13] the spectrum problem for $\widehat{P_k}$*-designs* $N(v, 2)$. In the following theorem we state the known results for $k = 2, 3, 4$.

Theorem 2 (Milici and Quattrocchi [13]). *For every* $v \geqslant 3$, *there is a* $\widehat{P_2}$*-design* $N(v, 2)$. *For every* $v \equiv 0$ or $1 \pmod 4$, $v \geqslant 4$, *there is a* $\widehat{P_3}$*-design* $N(v, 2)$. *For every* $v \equiv 0$ or $1 \pmod 3$, $v \geqslant 5$ *there is a* $\widehat{P_4}$*-design* $N(v, 2)$ *except possibly if* $v \in \{16, 39, 52, 70\}$.

Remark 1. We admit repeated blocks. So it will be sufficient for each G and each v to solve the spectrum problem only for the smallest positive λ such that a $2\lambda K_v \to \hat{G}$ can exist.

Generally, two well-known methods are used in construction: the difference method (see e.g. [6]) and the composition method (see e.g. [21,2,3]).

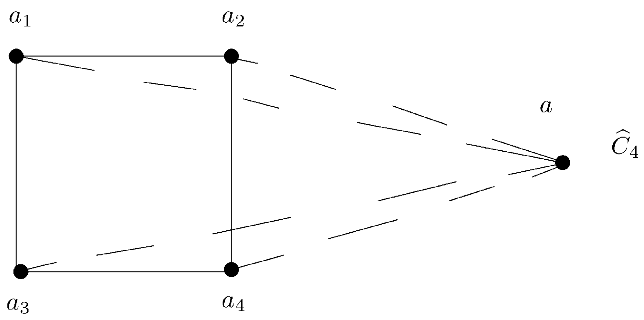

Fig. 6.

Usually, using the difference method, we will give only the *base blocks* of the decomposition as illustrated in the following examples.

Example 3. (3.1) The base block of the $\hat{D}$-design $N(9,2)$ given in Example (2.1) is $\langle 0,3,7 \bowtie 8;2\rangle \pmod 9$.

(3.2) The base blocks of the $\widehat{P}_4$-design $N(5,6)$ given in Example (2.2) are $\langle 0,1,3,\hat{2};4\rangle$ and $\langle 3,4,2,\hat{0};1\rangle \pmod 5$.

(3.3) The base block of the $\widehat{\widehat{P}}_4$-design $N(7,2)$ given in Example (2.3) is $\langle 5,\hat{1},6,0;4\rangle \pmod 7$.

(3.4) Put $V(K_{15}) = Z_5 \times Z_3$. The base blocks of the $\widehat{S}_3$-design $N(15,2)$ given in Example (2.4) are: $\langle \widehat{(0,0)};(1,0),(2,0),(0,1);(3,0)\rangle$, $\langle \widehat{(0,0)};(1,1),(2,1),(3,1);(2,0)\rangle$, $\langle \widehat{(0,1)};(1,0),(1,1),(3,1);(4,1)\rangle$, $\langle \widehat{(0,1)};(0,2),(1,2),(2,2);(1,1)\rangle$, $\langle \widehat{(0,2)};(1,2),(3,2),(0,0);(4,2)\rangle$, $\langle \widehat{(0,0)};(1,2),(2,2),(3,2);(1,0)\rangle$ and $\langle \widehat{(0,2)};(1,0),(2,1),(1,1);(4,2)\rangle$ $(\mathrm{mod}\,(5,-))$.

(3.5) Put $V(K_8) = Z_7 \cup \{\infty\}$. The base blocks of the $\widehat{\widehat{S}}_3$-design $N(8,6)$ given in Example (2.5) are: $\langle 0;\infty,1,\hat{3};2\rangle$, $\langle 0;\infty,1,\hat{2};3\rangle$, $\langle 0;\infty,3,\hat{2};4\rangle$ and $\langle 0;1,2,\hat{4};3\rangle \pmod 7$.

(3.6) Put $V(K_8) = Z_7 \cup \{\infty\}$. The base blocks of the $\widehat{E}_2$-design $N(8,2)$ given in Example (2.6) are: $\langle \infty,\hat{2};1,\hat{4};0\rangle$ and $\langle 0,\hat{1};5,\hat{3};2\rangle \pmod 7$.

(3.7) The base blocks of the $\widehat{\widehat{E}}_2$-design $N(5,4)$ given in Example (2.7) are: $\langle 0,1;\hat{2},\hat{4};3\rangle$ and $\langle 2,4;\hat{0},\hat{1};3\rangle \pmod 5$.

Let Y be a finite set of *points*, $\mathscr{C}$ a family of distinct subsets of Y called *groups* which partition Y, and $\mathscr{A}$ a collection of subsets of Y called *blocks*. Let v and λ be positive integers and K and M sets of positive integers. The triple $(Y,\mathscr{C},\mathscr{A})$ is called a *group divisible design* (GDD) $GD[K,\lambda,M;v]$ if:

(c_1) $|Y| = v$;
(c_2) $\{|C| \mid C \in \mathscr{C}\} \subseteq M$;
(c_3) $\{|B| \mid B \in \mathscr{A}\} \subseteq K$;
(c_4) $|C \cap B| \leqslant 1$ for every $C \in \mathscr{C}$ and every $B \in \mathscr{A}$;

(c_5) every pairset $\{x, y\} \subseteq Y$ such that x and y belong to distinct groups is contained in exactly λ blocks of $\mathscr{A}$.

If $\mathscr{C}$ contains t_i groups of size m_i, for $i = 1, 2, \ldots, s$, we call $m_1^{t_1} m_2^{t_2} \ldots m_s^{t_s}$ the group type of the GDD. When $K = \{k\}$ we will write $GD[k, \lambda, M; v]$ instead of $GD[\{k\}, \lambda, M; v]$.

A $GD[K, \lambda, \{1\}; v]$ with group type 1^v is called a *pairwise balanced design*, denoted by $(Y, \mathscr{A})$ or (v, K, λ)-PBD. A (v, k, λ)-PBD is simply a K_k-design.

A $GD[k, 1, \{m\}; km]$ is called a *transversal design*, denoted by $TD[k, m]$.

Let $2\lambda K_{n_1, n_2, \ldots, n_h}$ be the complete multipartite multigraph on vertices $\bigcup_{i=1}^{h} X_i$, $|X_i| = n_i$, with exactly 2λ edges joining each pair of vertices from different sets X_i, X_j, $i \neq j$. The composition method is based on the following lemmas.

Lemma 2. *Suppose there exist a $\hat{G}$-design $N(w + n_i, 2\lambda)$ containing a subdesign $N(w, 2\lambda)$ (it could be $w = 0, 1$), $i = 1, 2, \ldots, h$, and a $2\lambda K_{n_1, n_2, \ldots, n_h} \to \hat{G}$. Then there exists a $\hat{G}$-design $N(w + n_1 + n_2 + \cdots + n_h, 2\lambda)$.*

Lemma 3 (Bermond and Schonheim [2] and Bermond et al. [3]). *If $2K_{n,n,n} \to \hat{G}$ then $2K_{pn, pn, pn} \to \hat{G}$ for every positive integer p.*

Lemma 4 (Bermond et al. [3]). *If $2K_{n,n,n,n} \to \hat{G}$ then $2K_{pn, pn, pn, pn} \to \hat{G}$ for every positive integer $p \neq 2, 6$.*

Lemma 5 (Bermond et al. [3]). *If $2K_{n,n,n} \to \hat{G}$ and $2K_{n,n,n,n} \to \hat{G}$, then $2K_{pn, pn, pn, qn} \to \hat{G}$ for $p \neq 2, 6$, $0 \leqslant q \leqslant p$.*

Lemma 6 (Bermond et al. [3] and Todorov [18]). *If $2K_{n,n,n,n,n} \to \hat{G}$ then $2K_{pn, pn, pn, pn, pn} \to \hat{G}$ for every positive integer $p \neq 2, 3, 6, 10$.*

Lemma 7. *Let $(Y, \mathscr{A})$ be a (v, K, λ_1)-PBD. If there exists a $2\lambda_2 K_n \to \hat{G}$ for every $n \in K$, then there exists a $2\lambda_1 \lambda_2 K_v \to \hat{G}$.*

For a very complete survey about the existence of $(v, K, 1)$-PBD it is possible to see [1]. In the next lemma we report only the results we need in our proofs.

Lemma 8 (Bennett et al. [1]). *Let K, V, A, and B be the sets defined in Table 1. Then for every $v \in V$ and $v \notin A \cup B$ there exists $(v, K, 1)$-PBD. Note that the values in A are genuine exceptions whereas the values in B are possible exceptions.*

Table 1. Let $m = \min(K)$. In the sets A and B, nonnegative integers less than m are omitted, since 0 and 1 are always present and the remaining integers are always absent.

(1.1) $K = \{4, 5, 6, 7\}$, $V = \{v \in N \mid v \geqslant 4\}$, $A = \{8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$, $B = \emptyset$.
(1.2) $K = \{5, 6, 7, 8, 9\}$, $V = \{v \in N \mid v \geqslant 5\}$, $A = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28, 29, 32, 33, 34\}$, $B = \emptyset$.

(1.3) $K=\{5,7,8,9\}$, $V=\{v\in N \mid v\geqslant 5\}$, $A=\{6,10,11,12,13,14,15,16,17,18,19,20,22,23,24,26,27,28,29,30,31,32,33,34,38,39\}$, $B=\{42,43,44,46,51,52,60,94,95,96,98,99,100,102,104,106,107,108,110,111,116,138,139,140,142,143,146,150,154,156,158,162,163,166,167,170,172,173,174,206,228,243\}$.

(1.4) $K=\{5,8,9\}$, $V=\{v\in N \mid v\equiv 0 \text{ or } 1 \pmod 4\}$, $A=\{12,13,16,17,20,24,28,29,32,33,44\}$, $B=\{52,60,68,84,92,96,100,104,108,112,113,116,124,132,140,156,172,173,192,204,212,228,244,252,268,272,300,308,312\}$.

(1.5) $K=\{5,9,13\}$, $V=\{v\in N \mid v\equiv 1 \pmod 4\}$, $A=\{17,29,33\}$, $B=\{\emptyset\}$.

Lemma 9 (Truncation of groups of a transversal design [8]). *Let k be an integer, $k\geqslant 2$. Let $K=\{k,k+1,\dots,k+s\}$. Suppose that there exists a $TD[k+s,m]$. Let $g_1,g_2,\dots,g_s$ be integers satisfying $0\leqslant g_i\leqslant m$, $i=1,2,\dots,s$. Then there exists a GDD of type $m^k g_1 g_2\dots g_s$ with block sizes in K.*

Lemma 10. *Suppose there exists a $GD[t,1,M;v]$, a $2\lambda K_{n_1,n_2,\dots,n_t}\to\hat{G}$ (with $n_1=n_2=\dots=n_t=n$) and for any $m\in M$ a $\hat{G}$-design $N(mn+w,2\lambda)$ containing a subdesign $N(w,2\lambda)$ (it could be $w=0,1$). Then there exists a $\hat{G}$-design $N(nv+w,2\lambda)$.*

Example 4. Let $G=D$. Let $X_i=\{i,5+i\}i\in Z_5$, and $V(K_{2,2,2,2,2})=\bigcup_{i=0}^{4}X_i$. For a $2K_{2,2,2,2,2}\to\hat{G}$ take the base block $\langle 0,4,3\bowtie 5;1\rangle \pmod{10}$.

Put $w=1$ in Lemma 10. Since there exists a $GD[5,1,\{4\};20]$ [8] and a $N(9,2)$ (take $\langle 0,3,7\bowtie 8;2\rangle$ as base block), then Lemma 10 implies the existence of a $N(41,2)$.

Lemma 11. *Suppose there exists a (v,t,λ_1)-PBD, a $2\lambda_2 K_{n_1,n_2,\dots,n_t}\to\hat{G}$ (with $n_1=n_2=\dots=n_t=n$) and a $\hat{G}$-design $N((t-1)n+w,2\lambda_2)$ containing a subdesign $N(w,2\lambda_2)$ (it could be $w=0,1$). Then there exists a $\hat{G}$-design $N(nv+w,2\lambda_1\lambda_2)$.*

Lemma 12. *Suppose there exists: a $\hat{G}$-design (with $G\in\{P_3,P_4,C_4\}$) $N(v,2\lambda)$; a $\hat{G}$-design $N(w,2\lambda)$ containing a subdesign $N(q,2\lambda)$ (it could be $q=0,1$). Then there is a $\hat{G}$-design $N(v(w-q)+q,2\lambda)$.*

Proof. We prove the lemma only for $G=P_4$. Similarly it is possible to prove the remaining cases. Let $(Z_v,\mathscr{B})$ be a $\widehat{P_4}$-design $N(v,2\lambda)$. Let $(Z_{w-q},\cdot)$ be a quasigroup of order $w-q$. Let $T=\{\infty_1,\infty_2,\dots,\infty_q\}$ if $q>0$ and $T=\emptyset$ if $q=0$. Then define a $\widehat{P_4}$-design of order $v(w-q)+q$, $((Z_v\times Z_{w-q})\cup T,\mathscr{D})$ as follows:

(d_1) For every $\langle a,b,c,\hat{d};x\rangle\in\mathscr{B}$ put in $\mathscr{D}$ the following blocks $\langle (a,i),(b,j),(c,i),\widehat{(d,j)};(x,i\cdot j)\rangle$ for every $i,j\in Z_{w-q}$.

(d_2) For every $a\in Z_v$, put in $\mathscr{D}$ the blocks of a $N(w,2\lambda)$ on the point set $(\{a\}\times Z_{w-q})\cup T$ containing a subdesign $N(q,2\lambda)$ on the point set T. $\square$

Lemma 13. *Suppose there exist: a $\hat{D}$-*design $N(v,2\lambda)$; *a $\hat{D}$-design $N(w,2\lambda)$ containing a subdesign $N(q,2\lambda)$ (or $q=0,1$); two orthogonal quasigroups of order $w-q$. Then there is a $\hat{D}$-design $N(v(w-q)+q,2\lambda)$.*

Proof. Let $(Z_v, \mathscr{B})$ be a $\hat{D}$-design $N(v, 2\lambda)$. Let $(Z_{w-q}, \cdot)$ and $(Z_{w-q}, \circ)$ be two orthogonal quasi-groups of order $w - q$ (it is well-known [20] that these quasigroups exist for every $w - q \neq 2, 6$). Let $T = \{\infty_1, \infty_2, \ldots, \infty_q\}$ if $q > 0$ and $T = \emptyset$ if $q = 0$. Then define a $\hat{D}$-design of order $v(w - q) + q$, $((Z_v \times Z_{w-q}) \cup T, \mathscr{D})$ as follows:

(d_1) For every $\langle a, b, c \bowtie d; x\rangle \in \mathscr{B}$ put in $\mathscr{D}$ the blocks $\langle (a, i), (b, j), (c, i \cdot j) \bowtie (d, i); (x, i \circ j)\rangle, i, j \in Z_{w-q}$.
(d_2) For every $a \in Z_v$, put in $\mathscr{D}$ the blocks of a $N(w, 2\lambda)$ on the point set $(\{a\} \times Z_{w-q}) \cup T$ containing a subdesign $N(q, 2\lambda)$ on the point set T. $\square$

We complete this section by collecting some results for small values of v given in [14].

Theorem 3 (Millci and Quattrocchi [14]). *The following nested G-design are determined*:

$\widehat{P}_4$-*design* $N(16, 2)$;

$\widehat{P_3}$-*design* $N(v, 4)$ *for* $v = 6, 7, 10, 11, 14, 15, 18, 19, 23$;

$\widehat{P}_4$-*design* $N(v, 6)$ *for* $v = 5, 8, 11, 14, 17, 20, 23, 32$;

$\widehat{\overline{P_4}}$-*design* $N(v, 2)$ *for* $v = 9, 10, 15, 22, 34$;

$\widehat{\overline{P_4}}$-*design* $N(v, 6)$ *for* $v = 5, 8, 11, 14, 16, 17, 20, 23, 32$;

$\widehat{S}_3$-*design* $N(v, 2)$ *for* $v = 9, 10, 22$;

$\widehat{\overline{S_3}}$-*design* $N(v, 2)$ *for* $v = 9, 10, 15, 22$;

$\widehat{\overline{S_3}}$-*design* $N(v, 6)$ *for* $v = 5, 8, 11, 14, 17, 20, 23, 29, 32$;

$\widehat{C_4}$-*design* $N(v, 2)$ *for* $v = 65, 97, 113$;

$\widehat{C_4}$-*design* $N(v, 4)$ *for* $v = 5, 13, 21, 29$;

$\widehat{C_4}$-*design* $N(v, 16)$ *for* $v = 12, 24, 2^n$, $n \geqslant 3$;

$\hat{D}$-*design* $N(v, 2)$ *for* $v = 8, 16, 17, 24, 25, 32, 33, 56$;

$\hat{D}$-*design* $N(v, 4)$ *for* $v = 5, 12, 13, 28, 29, 52, 84$;

$\hat{D}$-*design* $N(v, 8)$ *for* $v = 7, 10, 11, 14, 15, 22, 23, 30, 34, 42$.

The following nested decompositions are determined:

$2K_{13,13,13} \to \widehat{P}_4$;

$2K_{13,13,13,13} \to \widehat{P}_4$;

$2K_{v,v,v} \to \widehat{\overline{P_4}}$, *for* $v = 3, 7, 13$;

$2K_{2,2,2,2} \to \widehat{\overline{P_4}}$;

$2K_{v,v,v} \to \widehat{S_3}$ *and* $2K_{v,v,v} \to \widehat{\widehat{S_3}}$, *for* $v = 3, 5, 7, 13$;

$2K_{v,v,v,v} \to \widehat{S_3}$ *and* $2K_{v,v,v,v} \to \widehat{\widehat{S_3}}$, *for* $v = 3, 10$.

3. Nesting of *G*-designs for *G*= P_2, P_3, E_2, P_4 and S_3

In this section we deal with the problem of constructing a nested G-design of order v for all the graphs G having four or less nonisolated vertices and at most three edges.

The spectrum of $\widehat{P_2}$-designs $N(v, 2\lambda)$ is an immediate consequence of Lemma 1, Theorem 2 and Remark 1.

Theorem 4. *The necessary and sufficient condition for the existence of a* $\widehat{P_2}$*-design* $N(v, 2\lambda)$ *is that* $v \geqslant 3$.

Theorem 5. *The necessary and sufficient condition for the existence of a* $\widehat{P_3}$*-design* $N(v, 2\lambda)$ *is that*:

(1) $v \equiv 0$ *or* $1 \pmod 4$, $v \geqslant 4$, *if* $\lambda \equiv 1 \pmod 2$.
(2) $v \geqslant 4$, *if* $\lambda \equiv 0 \pmod 2$.

Proof. The necessity follows from Lemma 1. Theorem 2 and Remark 1 get the sufficiency for odd λ. By composition method, Theorems 2 and 3, Lemmas 7 and 8 (Table (1.1)) we obtain the sufficiency for $\lambda = 2$. Remark 1 completes the proof. □

Theorem 6. *The necessary and sufficient condition for the existence of a* $\widehat{\widehat{P_3}}$*-design* $N(v, 2\lambda)$ *is that*:

(1) $v \equiv 0$ or $1 \pmod 4$, $v \geqslant 4$, *if* $\lambda \equiv 1 \pmod 2$.
(2) $v \geqslant 4$, *if* $\lambda \equiv 0 \pmod 2$.

Proof. The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda = 1$ if $v \equiv 0$ or $1 \pmod 4$ and for $\lambda = 2$ if $v \equiv 2$ or $3 \pmod 4$.

Let $v = 4h$, $h \geqslant 1$, and let $V(K_v) = Z_{v-1} \cup \{\infty\}$. The base blocks $(\mathrm{mod}\, 4h - 1)$ of a $\widehat{\widehat{P_3}}$-design $N(4h, 2)$, are: $\langle \infty, \hat{0}, 1; 2 \rangle$, and for $h \geqslant 2$, $\langle 2\rho + 2, \hat{0}, 2\rho + 3; 4\rho + 5 \rangle \rho = 0, 1, \ldots, h - 2$.

Let $v = 1 + 4h$, $h \geqslant 1$, and let $V(K_v) = Z_v$. The base blocks $(\mathrm{mod}\, 1 + 4h)$ of a $\widehat{\widehat{P_3}}$-design $N(4h + 1, 2)$ are: $\langle 2\rho + 1, \hat{0}, 2\rho + 2; 4\rho + 3 \rangle \rho = 0, 1, \ldots, h - 1$.

Let $v = 4h + 2$, $h \geqslant 1$, and let $V(K_v) = Z_{v-1} \cup \{\infty\}$. The base blocks $(\mathrm{mod}\, 1 + 4h)$ of a $\widehat{\widehat{P_3}}$-design $N(4h + 2, 4)$, are: $\langle \infty, \hat{0}, 1; 2 \rangle$, $\langle \infty, \hat{0}, 2; 4 \rangle$, $\langle 1, \hat{0}, 2; 3 \rangle$, and for $h \geqslant 2$, two copies of $\langle 2\rho + 3, \hat{0}, 2\rho + 4; 4\rho + 7 \rangle$, $\rho = 0, 1, \ldots, h - 2$.

Let $v = 4h + 3$, $h \geqslant 1$, and let $V(K_v) = Z_v$. The base blocks $(\mathrm{mod}\, 3 + 4h)$ of a $\widehat{\widehat{P_3}}$-design $N(4h + 3, 4)$, are: $\langle 1, \hat{0}, 2; 3 \rangle$, $\langle 1, \hat{0}, 3; 4 \rangle$, $\langle 2, \hat{0}, 3; 5 \rangle$, and for $h \geqslant 2$, two copies of $\langle 2\rho + 4, \hat{0}, 2\rho + 5; 4\rho + 9 \rangle$, $\rho = 0, 1, \ldots, h - 2$. □

Theorem 7. *The necessary and sufficient condition for the existence of a $\widehat{E_2}$-design $N(v,2\lambda)$ is that*:

(1) $v \equiv 0$ or $1 \pmod 4$, $v \geqslant 5$, *if* $\lambda \equiv 1 \pmod 2$.
(2) $v \geqslant 5$, if $\lambda \equiv 0 \pmod 2$.

Proof. The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda = 1$ if $v \equiv 0$ or $1 \pmod 4$ and for $\lambda = 2$ if $v \equiv 2$ or $3 \pmod 4$.

Let $v = 4h$, $h \geqslant 2$. Let $V(K_v) = Z_{v-1} \cup \{\infty\}$. The base blocks $(\mathrm{mod}\, 4h-1)$ are: $\langle \infty, \hat{2}; 1, \widehat{2h}; 0\rangle$, $\langle 0, \widehat{2\rho-1}; 4\rho+1, \widehat{2\rho+1}; 2\rho\rangle$ $\rho = 1,2,\ldots,h-1$.

Let $v = 1+4h$, $h \geqslant 1$. Let $V(K_v) = Z_v$. The base blocks $(\mathrm{mod}\, 1+4h)$ are: $\langle 0, \widehat{2\rho}; 4\rho-1, \widehat{6\rho-2}; 2\rho-1\rangle \rho = 1,2,\ldots,h$.

Let $v = 2+4h$, $h \geqslant 1$. Take $V(K_v) = Z_{v-1} \cup \{\infty\}$ and base blocks $(\mathrm{mod}\, 1+4h)$: $\langle \infty, \hat{2}; 1, \hat{3}; 0\rangle$, $\langle \infty, \hat{1}; 2, \hat{4}; 0\rangle$, $\langle 0, \hat{4}; 3, \hat{2}; 1\rangle$, and, for $h \geqslant 2$, two copies of $\langle 0, \widehat{2\rho+2}; 4\rho+3, \widehat{6\rho+4}; 2\rho+1\rangle$, $\rho = 1,2,\ldots,h-1$.

Let $v = 3+4h$, $h \geqslant 1$. Take $V(K_v) = Z_v$ and base blocks $(\mathrm{mod}\, 3+4h)$: $\langle 0, 4\widehat{h+2}; 2, \hat{3}; 1\rangle$, two copies of $\langle 0, \widehat{2h}; 2+4h, \widehat{1+2h}; 2+2h\rangle$, and for $h \geqslant 2$, two copies of $\langle 0, \widehat{2\rho+1}; 4\rho+1, \widehat{6\rho+1}; 2\rho\rangle$, $\rho = 1,2,\ldots,h-1$. □

Theorem 8. *The necessary condition for the existence of a $\widehat{\widehat{E_2}}$-design $N(v,2\lambda)$ is that*:

(1) $v \equiv 0$ *or* $1 \pmod 4$, $v \geqslant 5$, *if* $\lambda \equiv 1 \pmod 2$.
(2) $v \geqslant 5$, *if* $\lambda \equiv 0 \pmod 2$.

This necessary condition is also sufficient except possibly for $v = 5$ if $\lambda \equiv 1 \pmod 2$.

Proof. The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda = 1$ if $v \equiv 0$ or $1 \pmod 4$ and for $\lambda = 2$ if $v \equiv 2$ or $3 \pmod 4$.

Let $v = 4h$, $h \geqslant 2$. Take $V(K_v) = Z_{v-1} \cup \{\infty\}$ and the following base blocks $(\mathrm{mod}\, 4h-1)$: $\langle \infty, 2; \hat{1}, \widehat{2h}; 3\rangle$, $\langle 0, 2h-3; \hat{2}, \widehat{2h}; 2h-1\rangle$, and for $h \geqslant 3$, $\langle 0, 2\rho-1; \hat{1}, \widehat{2\rho+1}; \rho+h\rangle$, $\rho = 1,2,\ldots,h-2$.

Let $v = 1+4h$, $h \geqslant 2$. Take $V(K_v) = Z_v$. and base blocks $(\mathrm{mod}\, 1+4h)$:

If $h \not\equiv 1 \pmod 3$ and $h \not\equiv 3 \pmod 5$, then the base blocks $(\mathrm{mod}\, 1+4h)$ are:

$\langle 4\rho-1, 6\rho-2; \hat{0}, \widehat{2\rho}; 5\rho-h-2\rangle$, $\rho = 1,2,\ldots,h$.

If $h = 1+3\alpha$, $\alpha \geqslant 1$, then the base blocks $(\mathrm{mod}\, 1+4h)$ are:

$\langle 4\rho-1, 6\rho-2; \hat{0}, \widehat{2\rho}; 5\rho-h-2\rangle$, for $\rho \in \{1,2,\ldots,h\} - \{1+\alpha\}$, and
$\langle 3+4\alpha, 4+6\alpha; \hat{0}, 2\widehat{+2}\alpha; 5+8\alpha\rangle$;

If $h = 3+5\alpha$, $\alpha \geqslant 1$, then the base blocks $(\mathrm{mod}\, 1+4h)$ are:

$\langle 4\rho-1, 6\rho-2; \hat{0}, \widehat{2\rho}; 5\rho-h-2\rangle$, for $\rho \in \{1,2,\ldots,h\} - \{1+\alpha\}$, and
$\langle 3+4\alpha, 4+6\alpha; \hat{0}, 2\widehat{+2}\alpha; 7+10\alpha\rangle$.

Let $v = 2 + 4h$, $h \geqslant 2$. Take $V(K_v) = Z_{v-1} \cup \{\infty\}$ and base blocks (mod $1 + 4h$): $\langle \infty, 0; \hat{1}, \hat{2}; 2h \rangle$, $\langle \infty, 0; \hat{1}, \hat{2}; 2h-1 \rangle$, $\langle 0, 2; \hat{1}, \hat{3}; 1+2h \rangle$, and 2 copies of $\langle 0, 1+2\rho; \hat{1}, 3 \widehat{+ 2}\rho; 1+\rho \rangle$ for odd $\rho \in \{1, 2, \ldots, h-1\}$, $\langle 1, 3+2\rho; \hat{0}, 1 \widehat{+ 2}\rho; \rho + 2h + 2 \rangle$, for even $\rho \in \{1, 2 \ldots, h-1\}$.

Let $v = 3 + 4h$, $h \geqslant 1$. Take $V(K_v) = Z_v$ and base blocks, (mod $3 + 4h$): $\langle 1, 3 + 4h; \hat{2}, \hat{3}; 2 + 2h \rangle$, and 2 copies of $\langle 1, 2 + 2\rho; \hat{0}, \widehat{2\rho}; 2 + \rho \rangle$ for odd $\rho \in \{1, 2, \ldots, h\}$, $\langle 0, 2\rho; \hat{1}, 2 \widehat{+ 2}\rho; \rho + 2h + 2 \rangle$, for even $\rho \in \{1, 2, \ldots, h\}$.

To complete our proof note that a $\widehat{\widehat{E_2}}$-design $N(5,4)$ is given by Example (3.7) and a $\widehat{\widehat{E_2}}$-design $N(6,4)$ is the following: $V(K_6) = Z_5 \cup \{\infty\}$ and the base blocks (mod 5) are $\langle \infty, 0; \hat{1}, \hat{2}; 3 \rangle$, $\langle \infty, 0; \hat{1}, \hat{2}; 4 \rangle$, $\langle 0, 2; \hat{1}, \hat{3}; 4 \rangle$. $\square$

Remark 2. It is easy to verify that there is not a $\widehat{\widehat{E_2}}$-design $N(5, 2\lambda)$ for $\lambda = 1, 3$. But we are unable to prove the nonexistence of these designs for every odd λ.

Theorem 9. *The necessary and sufficient condition for the existence of a $\widehat{P_4}$-design $N(v, 2\lambda)$ is that*:

(1) $v \equiv 0$ or $1\,(\mathrm{mod}\,3)$, $v \geqslant 6$, *if* $\lambda \equiv 1$ or $2\,(\mathrm{mod}\,3)$.
(2) $v \geqslant 5$, if $\lambda \equiv 0\,(\mathrm{mod}\,3)$.

Proof. The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda = 1$ if $v \equiv 0$ or $1\,(\mathrm{mod}\,3)$ and for $\lambda = 3$ if $v \equiv 2\,(\mathrm{mod}\,3)$.

Let $v \equiv 0$ or $1\,(\mathrm{mod}\,3)$, then the sufficiency follows from Theorem 2 except possibly for $v = 16, 39, 52, 70$. By Theorem 3 there is a $\widehat{P_4}$-design $N(16,2)$. A $\widehat{P_4}$-design $N(v,2)$ for $v = 39, 52$ is given by Lemma 2 where we put $w = 0$, $n_i = 13$ and $h = 3, 4$ respectively (a $\widehat{P_4}$-design $N(13,2)$ there is by Theorem 2 and the decompositions $2K_{13,13,13} \to \widehat{P_4}$ and $2K_{13,13,13,13} \to \widehat{P_4}$ by Theorem 3). Lemma 12 with $v = 10$, $w = 7$ and $q = 0$ implies the existence of a $\widehat{P_4}$-design $N(70,2)$.

Let $v \equiv 2\,(\mathrm{mod}\,3)$. The existence of a $\widehat{P_4}$-design $N(29,6)$ follows from Lemma 7 since there is a decomposition $3K_{29} \to K_7$ ([12]) and a $\widehat{P_4}$-design $N(7,2)$ (Theorem 2). The remaining cases follow from Theorem 3, the composition method, Lemma 7 and Lemma 8 (Table (1.2)). $\square$

Theorem 10. *The necessary condition for the existence of a $\widehat{\widehat{P_4}}$-design $N(v, 2\lambda)$ is that*:

(1) $v \equiv 0$ or $1\,(\mathrm{mod}\,3)$, $v \geqslant 5$, *if* $\lambda \equiv 1$ or $2\,(\mathrm{mod}\,3)$.
(2) $v \geqslant 5$, *if* $\lambda \equiv 0\,(\mathrm{mod}\,3)$.

This necessary condition is also sufficient except possibly for $v = 52$ if $\lambda \equiv 1$ or $2\,(\mathrm{mod}\,3)$.

Proof. The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda = 1$ if $v \equiv 0$ or $1\,(\mathrm{mod}\,3)$ and for $\lambda = 3$ if $v \equiv 2\,(\mathrm{mod}\,3)$.

Suppose at first $v \equiv 0$ or $1 \pmod 3$. Let $v = 6h$, $h \geqslant 1$. Take $V(K_v) = Z_{v-1} \cup \{\infty\}$ and base blocks $(\bmod\ 6h-1)$: $\langle 1, \widehat{6h-2}, 0, \infty; 3h-1 \rangle$, and if $h \geqslant 2$, $\langle 6h-2-3\rho, \hat{1}, 6h-1-3\rho, 0; 6h-2-3(\rho-1)/2 \rangle$, for odd $\rho \in \{1,2,\dots,h-1\}$, $\langle 6h-2-3\rho, \hat{1}, 6h-1-3\rho, 0; 3h-3\rho/2 \rangle$, for even $\rho \in \{1,2,\dots,h-1\}$.

Let $v = 1+6h$, $h \geqslant 1$. Take $V(K_v) = Z_v$ and base blocks $(\bmod\ 6h+1)$: $\langle 6h-1-3\rho, \hat{1}, 6h-3\rho, 0; 4h-\rho \rangle$, $\rho = 0,1,\dots,h-1$.

Let $v \equiv 3$ or $4 \pmod 6$. A $\widehat{\overline{P_4}}$-design $N(21,2)$ is obtained by Lemma 2 with $w=0$, $n_i = 7$ and $h=3$ (a $2K_7 \to \widehat{\overline{P_4}}$ is given in Example (3.3) and a $2K_{7,7,7} \to \widehat{\overline{P_4}}$ there is by Theorem 3). A $\widehat{\overline{P_4}}$-design $N(33,2)$ can be constructed by Lemma 10 with $t=4$, $v=16$, $n=2$, $w=1$ and $m=4$ (it is well-known that there is a $GD[4,1,\{4\};16]$, and a $2K_{2,2,2,2} \to \widehat{\overline{P_4}}$ there is by Theorem 3). A $\widehat{\overline{P_4}}$-design $N(39,2)$ is obtained by Lemma 2 with $w=0$, $n_i=13$ and $h=3$ (a $2K_{13,13,13} \to \widehat{\overline{P_4}}$ there is by Theorem 3). The existence of a $\widehat{\overline{P_4}}$-design $N(40,2)$ follows from Lemma 4 with $n=2$ and $p=5$ and Lemma 2 with $w=0$, $n_i=10$ and $h=4$.

The sufficiency for the remaining values of $v \geqslant 6$, $v \equiv 3$ or $4 \pmod 6$, $v \neq 52$, follow from Theorem 3, Lemma 10 (with $w=0$ if $v \equiv 3 \pmod 6$ and $w=1$ if $v \equiv 4 \pmod 6$), and the existence of a $2K_{3,3,3} \to \widehat{\overline{P_4}}$, and the following GDDs [5]: $GD[3,1,\{3,5\};23]$, $GD[3,1,\{3\};3+6\alpha]$ for any $\alpha \geqslant 1$, $GD[3,1,\{3,7\};7+6\alpha]$ for any $\alpha \geqslant 2$ and $GD[3,11,\{3\};11+6\alpha]$ for any $\alpha \geqslant 1$.

Now suppose $v \equiv 2 \pmod 3$, $v \geqslant 3$. For $v=29$ the proof follows from Lemma 7 and the existence of a decomposition $3K_{29} \to K_7$ ([12]) and the $\widehat{\overline{P_4}}$-designs $N(7,2)$ above constructed.

The composition method, Lemma 7, Lemma 8 (Table (1.2)) and Theorem 3 complete the proof. □

Theorem 11. *The necessary and sufficient condition for the existence of a $\widehat{S_3}$-design $N(v, 2\lambda)$ is that*:

(1) $v \equiv 0$ or $1 \pmod 3$, $v \geqslant 6$, *if* $\lambda \equiv 1$ or $2 \pmod 3$.
(2) $v \geqslant 5$, *if* $\lambda \equiv 0 \pmod 3$.

Proof. The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda = 1$ if $v \equiv 0$ or $1 \pmod 3$ and for $\lambda = 3$ if $v \equiv 2 \pmod 3$.

Suppose at first $v \equiv 0$ or $1 \pmod 6$. If $v = 6h$, $h \geqslant 1$, then take $V(K_v) = Z_{v-1} \cup \{\infty\}$ and base blocks, $(\bmod\ 6h-1)$: $\langle \hat{0}; \infty, 1, 2; 3 \rangle$ and, if $h \geqslant 2$, $\langle \hat{0}; 3\rho+3, 3\rho+4, 3\rho+5; 6\rho+8 \rangle$, $\rho = 0,1,\dots,h-2$.

If $v = 1+6h$, $h \geqslant 1$, then take $V(K_v) = Z_v$ and base blocks, $(\bmod\ 1+6h)$, $\langle \hat{0}; 3\rho+1, 3\rho+2, 3\rho+3; 6\rho+4 \rangle$, $\rho = 0,1,\dots,h-1$.

Let now $v \equiv 3$ or $4 \pmod 6$. The sufficiency for $v = 9, 10, 22$ is given in Theorem 3. For $v = 15$ see the Example (3.4). It is easy to apply Lemmas 2, 3, 4 and Theorem 3 to prove the sufficiency for $v = 16, 21, 27, 28, 39, 40, 63$. As an example we prove the case $v = 16$ leaving the remaining ones to the reader: Lemma 3 (with $n=5$, $p=3$ and the $2K_{5,5,5} \to \widehat{S_3}$ given in Theorem 3) gets a $2K_{15,15,15} \to \widehat{S_3}$. Then Lemma 2 with $w=1$, $h=3$ and $n_i=15$ completes the proof.

Lemmas 3 and 5 imply a $2K_{3p,3p,3p} \to \widehat{S_3}$ (for any p) and a $2K_{3p,3p,3p,3q} \to \widehat{S_3}$ (for $p \neq 2, 6$, $0 \leqslant q \leqslant p$). Therefore by Lemma 2 we obtain that if for $w=0,1$ there is a $2K_{3p+w} \to \widehat{S_3}$ (or a $2K_{3p+w} \to \widehat{S_3}$ and a $2K_{3q+w} \to \widehat{S_3}$) then there exists a $2K_{9p+w} \to \widehat{S_3}$ (or a $2K_{9p+3q+w} \to \widehat{S_3}$, $p \neq 2, 6$, respectively). By induction, starting with the $\widehat{S_3}$-designs $N(v,2)$ for $v=16, 21, 27, 28, 39, 40, 63$, we complete the proof for $v \equiv 3$ or $4 \pmod 6$. As an example we prove the sufficiency for $v=51, 52$. Let $p=5$ and $q=2$, then there is a $2K_{15,15,15,6} \to \widehat{S_3}$. A $2K_6 \to \widehat{S_3}$ is given above and a $2K_{15} \to \widehat{S_3}$ there is by Theorem 3. Then we obtain a $2K_{51} \to \widehat{S_3}$ (for $w=0$) and a $2K_{52} \to \widehat{S_3}$ (for $w=1$).

At last we prove the sufficiency for $v \equiv 2 \pmod 3$.

Let $v=5+6h$, $h \geqslant 0$. Take $V(K_v)=Z_v$ and base blocks (mod $5+6h$): $\langle \hat{0}; 1, 2+3h, 3+6h; 4+6h\rangle$, $\langle \hat{0}; 1+3h, 2+3h, 4+6h; 3+6h\rangle$, and, for $h \geqslant 1$, $\langle \hat{0}; 3\rho+1, 3\rho+2, 3\rho+3; 6\rho+4\rangle$, $\langle \hat{0}; 3\rho+2, 3\rho+3, 3\rho+4; 6\rho+6\rangle$, $\langle \hat{0}; 3\rho+3, 3\rho+4, 3\rho+5; 6\rho+8\rangle$, $\rho=0,1,\ldots,h-1$.

Let $v=2+6h$, $h \geqslant 1$. Take $V(K_v)=Z_{v-1} \cup \{\infty\}$ and base blocks (mod $1+6h$): $\langle \hat{0}; \infty, 1, 2; 3\rangle$, $\langle \hat{0}; \infty, 1, 3; 4\rangle$, $\langle \hat{0}; \infty, 2, 3; 5\rangle$, $\langle \hat{0}; 1, 2, 3; 4\rangle$ and, for $h \geqslant 2$, three copies of $\langle \hat{0}; 3\rho+4, 3\rho+5, 3\rho+6; 6\rho+10\rangle$, $\rho=0,1,\ldots,h-2$. □

Theorem 12. *The necessary and sufficient condition for the existence of a* $\widehat{\widehat{S_3}}$-design $N(v, 2\lambda)$ *is that*:

(1) $v \equiv 0$ or $1 \pmod 3$, $v \geqslant 6$, *if* $\lambda \equiv 1$ or $2 \pmod 3$.
(2) $v \geqslant 5$, if $\lambda \equiv 0 \pmod 3$.

Proof. The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda=1$ if $v \equiv 0$ or $1 \pmod 3$ and for $\lambda=3$ if $v \equiv 2 \pmod 3$.

Suppose at first $v \equiv 0$ or $1 \pmod 6$. Let $v=6h$, $h \geqslant 1$. Take $V(K_v)=Z_{v-1} \cup \{\infty\}$ and base blocks (mod $6h-1$):

$\langle 0; \infty, 2, \hat{1}; 3h\rangle$ and, if $h \geqslant 2$,
$\langle 0; 3\rho, 3\rho+1, \widehat{3\rho+2}; \frac{3\rho-1}{2}\rangle$ for odd $\rho \in \{1, 2, \ldots, h-1\}$,
$\langle 0; 3\rho, 3\rho+1, \widehat{3\rho+2}; 3h-1+\frac{3\rho}{2}\rangle$ for even $\rho \in \{1, 2, \ldots, h-1\}$.

Let $v=1+6h$, $h \geqslant 1$. Take $V(K_v)=Z_v$ and base blocks (mod $1+6h$):
$\langle 0; 2, 3, \hat{1}; 3h+1\rangle$ and, if $h \geqslant 2$,
$\langle 0; 3\rho+1, 3\rho+2, \widehat{3\rho+3}; \frac{3(\rho+1)}{2}\rangle$ for odd $\rho \in \{1, 2, \ldots, h-1\}$,
$\langle 0; 3\rho+2, 3\rho+3, \widehat{3\rho+1}; 3h+1+\frac{3\rho}{2}\rangle$ for even $\rho \in \{1, 2, \ldots, h-1\}$.

Let now $v \equiv 3$ or $4 \pmod 6$. The sufficiency for $v=9, 10, 15, 22$ follows from Theorem 3. It is easy to apply Lemmas 2, 3, 4 and Theorem 3 to prove the sufficiency for $v=16, 21, 27, 28, 39, 40, 63$. As an example we prove the case $v=16$ leaving the remaining ones to the reader: Lemma 3 (with $n=5$, $p=3$ and the $2K_{5,5,5} \to \widehat{S_3}$ given in Theorem 3) gets a $2K_{15,15,15} \to \widehat{\widehat{S_3}}$. Then Lemma 2 with $w=1$, $h=3$ and $n_i=15$ completes the proof.

To complete the proof of the sufficiency when $v \equiv 3$ or $4 \pmod 6$ proceed as in the above Theorem 11.

By Lemma 8 (Table (1.2)) and Theorem 3 it follows the proof for $\lambda = 3$ and $v \equiv 2 \pmod 3$, $v \geqslant 5$. □

4. Nesting of *G*-designs for $G = C_4$ and *D*

In this section we deal with the problem of constructing a nested *G*-design of order v for all the graphs G having four nonisolated vertex and four edges.

Theorem 13. *The necessary condition for the existence of a* $2\lambda K_v \to \widehat{C_4}$ *is that*:

(1) $v \equiv 1 \pmod 8$, $v \geqslant 9$, *if* $\lambda \equiv 1 \pmod 2$;
(2) $v \equiv 1 \pmod 4$, $v \geqslant 5$, *if* $\lambda \equiv 2 \pmod 4$;
(3) $v \equiv 1 \pmod 2$, $v \geqslant 5$, *if* $\lambda \equiv 4 \pmod 8$;
(4) $v \geqslant 5$, *if* $\lambda \equiv 0 \pmod 8$.

Proof. There is not a nested C_4-design $(V, \mathscr{B})$ of order $v \equiv 0 \pmod 4$ and index $\lambda \equiv 2 \pmod 4$.

Suppose a such nested C_4-design existed. Let $(V, \mathscr{S})$ be the associated S_4-design. Put $\lambda = 4\rho + 2$ and $v = 4h$. Then the number of 4-cycles of $\mathscr{B}$ meeting the same vertex $a \in V$ is given by $(2\rho + 1)(4h - 1)$. Clearly a appears as terminal vertex of $(2\rho + 1)(4h - 1)$ 4-stars of $\mathscr{S}$. Let x be the number of 4-stars of $\mathscr{S}$ containing a as a centre. Therefore it is $4x + (2\rho + 1)(4h - 1) = (4\rho + 2)(4h - 1)$. This equality is impossible.

Similarly it is possible to prove that there is not a nested C_4-design of order $v \equiv 0 \pmod 2$ and index $\lambda \equiv 4 \pmod 8$.

Lemma 1 completes the proof. □

Theorem 14. *The necessary condition for the existence of a* $\widehat{C_4}$*-design* $N(v, 2\lambda)$ *given in the above Theorem* 13 *is also sufficient except possibly if* $v = 57, 185, 265$ *and* $\lambda \equiv 1 \pmod 2$.

Proof. By Remark 1 it is enough to prove the sufficiency for the smallest values of λ.

Case (1). Let $v \equiv 1 \pmod 8$, $v \geqslant 9$ and $\lambda = 1$. The proof follows from Theorem 1 and 3.

Case (2). Let $v \equiv 1 \pmod 4$, $v \geqslant 5$ and $\lambda = 2$. By Case (1), Lemma 8 (Table (1.5)) and Theorem 3 we obtain the proof.

Case (3). Let $v \equiv 1 \pmod 2$ and $\lambda = 4$. A $\widehat{C_4}$-design $N(v, 8)$ is given by $V(K_v) = Z_v$ and base blocks $\langle x, 2x, v - x, v - 2x; 0\rangle$, $x = 1, 2, \ldots, (v - 1)/2$, $(\mathrm{mod}\ v)$.

Case (4). Let $v \geqslant 5$ and $\lambda = 8$.

We start by proving the sufficiency for $v = 6, 10, 14, 18, 22, 26, 30, 34$. Suppose it is possible to construct by the difference method a $\widehat{C_4}$-design $N(w, 2\lambda)$ $(V, \mathscr{B})$ with $\lambda \in \{1, 2\}$. Let $\mathscr{B}^*$ be the set of base blocks of $\mathscr{B}$. Suppose $\infty \notin V$ and $\langle x, y, z, t; u\rangle \in \mathscr{B}^*$. If $\lambda = 1$ then the base blocks of a $\widehat{C_4}$-design $N(w + 1, 8)$ are:

$\langle\infty,x,y,z;u\rangle$, $\langle\infty,y,z,t;u\rangle$, $\langle\infty,z,t,x;u\rangle$, $\langle\infty,t,x,y;u\rangle$, $\langle x,y,z,t;\infty\rangle$, $9-4\lambda$ copies of $\langle x,y,z,t;u\rangle$ and $12-4\lambda$ copies of each $b\in\mathscr{B}^*$, $b\neq\langle x,y,z,t;u\rangle$.

Therefore it is sufficient to construct, by difference method, a $\widehat{C_4}$-design $N(w,2\lambda)$ with $w\in\{5,9,13,17,21,25,29,33\}$ and either $\lambda=1$ or $\lambda=2$.

For $w\in\{5,13,29\}$ the existence of a such $\widehat{C_4}$-design $N(w,4)$ is proved in Case (2).

Base blocks of a $\widehat{C_4}$-design $N(w,2)$ for $w\in\{9,17,25,33\}$ are found in [16] and for $w=21$ in Theorem 3.

The sufficiency for $v=2^n$, $n\geqslant 3$, $v=12$ and $v=24$ is given by direct construction in Theorem 3.

Cases $v=20,28$ follow from Lemma 7, the existence of the decompositions ([12]) $4K_{20}\to K_5$, $2K_{28}\to K_7$ and that of the $\widehat{C_4}$-designs $N(5,4)$ and $N(7,8)$.

By above results, Lemma 7 and Lemma 8 (Table (1.2)) we obtain the proof. □

Theorem 15. *The necessary condition for the existence of a $\hat{D}$-design $N(v,2\lambda)$ is that*:

(1) $v\equiv 0$ or $1\,(\mathrm{mod}\,8)$, $v\geqslant 8$, if $\lambda\equiv 1\,(\mathrm{mod}\,2)$.
(2) $v\equiv 0$ or $1\,(\mathrm{mod}\,4)$, $v\geqslant 5$, if $\lambda\equiv 2\,(\mathrm{mod}\,4)$.
(3) $v\geqslant 5$ if $\lambda\equiv 0\,(\mathrm{mod}\,4)$.

This necessary condition is also sufficient except possibly for $v=124,212$ if $\lambda\equiv 2\,(\mathrm{mod}\,4)$ and $v=6$ if $\lambda\equiv 0\,(\mathrm{mod}\,4)$.

Proof. The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda=1$ if $v\equiv 0$ or $1\,(\mathrm{mod}\,8)$, for $\lambda=2$ if $v\equiv 4$ or $5\,(\mathrm{mod}\,8)$ and for $\lambda=4$ if $v\equiv 2$ or $3\,(\mathrm{mod}\,4)$.

Case (1). Let $v\equiv 0$ or $1\,(\mathrm{mod}\,8)$ and $\lambda=1$.

A $\hat{D}$-design $N(9,2)$ is given in Example (3.1) and a $\hat{D}$-design $N(v,2)$ for $v=8,16,17,24,25,32,33,56$ there is by Theorem 3.

The existence of a $\hat{D}$-design $N(v,2)$ for every admissible v except possibly if $v=57,64,65$ follows from Lemma 6 (a $2K_{2,2,2,2,2}\to\hat{D}$ is given in Example 4), Lemma 10, and the existence of following GDDs [7,8,19]: a $GD[5,1,\{8\};48]$; a $GD[5,1,\{4\};v]$ for every $v\equiv 0$ or $4\,(\mathrm{mod}\,20)$; a $GD[5,1,\{4,8\};v]$ for every $v\equiv 8$ or $16\,(\mathrm{mod}\,20)$, $v\geqslant 36$, except possibly if $v=48$; a $GD[5,1,\{4,12\};v]$ for every $v\equiv 12\,(\mathrm{mod}\,20)$, $v\geqslant 52$.

To prove the cases $v=57,64,65$ use Lemma 13.

Case (2). Let $v\equiv 4$ or $5\,(\mathrm{mod}\,8)$, $v\geqslant 5$ $v\neq 124,212$ and $\lambda=2$.

By the above Case (1), Theorem 3, Lemma 7 and Lemma 8 (Table (1.4)), there is a $\hat{D}$-design $N(v,4)$ for every admissible $v\notin\{20,44,60,68,92,100,108,116,124,132,140,156,172,173,204,212,228,244,252,268,300,308\}$.

The cases $v=20,60,68,100,108,140,228,268,300,308$ follow from Lemma 11 and the existence of a $2K_v\to K_5$ for every $v\equiv 1$ or $5\,(\mathrm{mod}\,10)$, $v\neq 15$ [8].

The cases $v=44,173$ follow from Lemma 7 and the existence of a $(45,2,9)$-PBD [12] and a $(173,1,\{5,13\})$-PBD [7].

The cases $v=92,116,156,172,204,244$ follow from Lemma 13.

The cases $v = 132, 252$ follow from Lemma 10 and the existence of a $GD[5, 1, \{6\}; v]$ for $v = 66, 126$, [19].

Case (3). Let $v \equiv 2$ or $3 \pmod 4$, and $\lambda = 4$.

For $v = 7, 10, 11, 14, 15, 22, 23, 30, 34, 42$ a $\hat{D}$-design $N(v, 8)$ there is by Theorem 3.

For any $v \in \{94, 95, 98, 99, 110, 138, 139, 142, 143, 146, 150, 154, 162, 163, 170, 172, 243\}$ there is a $(v, 1, \{8, 9, 10\})$-PBD [1]. Since for $v = 5, 7, 8, 9$ there is a $N(v, 8)$, then by Lemma 7 and Lemma 8 (Table (1.3)) there exists a $\hat{D}$-design $N(v, 8)$ for each $v \equiv 2$ or $3 \pmod 4$ such that $v \notin \{6, 18, 19, 26, 27, 31, 38, 39, 43, 46, 51, 102, 106, 107, 111, 158, 166, 167, 174, 206\}$.

By truncation of groups of the transversal design $TD[17, 16]$ (Lemma 9) construct a GDD of type $10^4 9^6 16^7$. Then by Lemma 7 there is a $\hat{D}$-design $N(206, 8)$. Similarly we can prove the theorem for $v = 102, 107, 158, 167, 174$.

By Lemma 11 with $\lambda_1 = 4$ and $\lambda_2 = 1$ and the existence of a $(20, 5, 4)$-PBD [12], we obtain the proof for $v = 38, 39$.

The cases $v = 18, 19, 26, 27, 31, 43$ follow from lemma 7 and the existence of a $(19, 9, 4)$-PBD, a $(27, 9, 4)$-PBD, $(31, 5, 2)$-PBD and a $(43, 8, 4)$-PBD.

To prove cases $v = 46, 51, 106, 111, 166$ use Lemma 13. □

Remark 3. It is easy to verify that there is not a $\hat{D}$-design $N(6, 2\lambda)$ for $\lambda = 4$. But we are unable to prove the nonexistence of these designs for every $\lambda \equiv 0 \pmod 4$, $\lambda \geqslant 8$.

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